

# Counting endomorphisms of crown-like orders

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### **Abstract**

The authors introduce the notion of crown-like orders and introduce powerful tools for counting the endomorphisms of orders of this type.

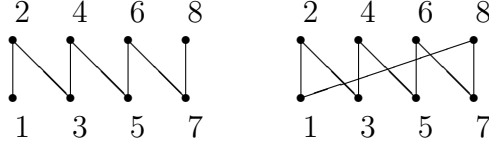


Figure 1: The fence and crown of order 4

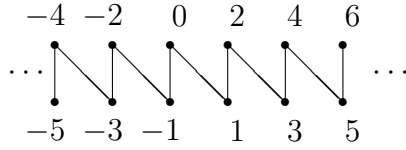


Figure 2: The infinite fence  $\mathcal{F}$

## 1 Introduction

Let  $n$  be a natural number. By the **fence of order**  $n$  we mean the order  $\mathcal{F}_n$  on  $\{1, 2, \dots, 2n\}$  where the odd elements are minimal, the even elements are maximal, and elements  $i$  and  $j$  are comparable exactly when  $i$  and  $j$  differ by 1. The **crown of order**  $n$  is the order  $\mathcal{C}_n$  on  $\{1, 2, \dots, 2n\}$  where the odd elements are minimal, the even elements are maximal, and elements  $i$  and  $j$  are comparable exactly when  $i$  and  $j$  differ by 1 modulo  $2n$ . (See Figure 1.)

Crowns and fences are two types of orders for which the number of endomorphisms is known exactly [1, 2]. Duffus et al. assert that the methods of [2] should extend to give asymptotic estimates for the number of endomorphisms of the orders depicted in Figure 3.

One sees that the orders in Figure 3 are ‘fence-like’ in their repetition of a certain basic unit. In this paper we propose a definition for fence-like and crown-like orders which will include the orders of Figure 3 as special cases. We also introduce a powerful method of exactly counting the endomorphisms of crown-like orders.

## 2 Generalized Crowns

In [2] it turns out to be useful to consider the **infinite fence**. This is the order  $\mathcal{F}$  on  $\mathbb{Z}$  where the odd elements are minimal, the even elements are maximal, and elements  $i$  and  $j$  are comparable exactly when  $i$  and  $j$  differ by 1. (See Figure 2.) This order has the shift map  $\sigma_2 : \mathbb{Z} \rightarrow \mathbb{Z}$  given by  $\sigma_2(i) = i + 2$  as an automorphism. The restriction of  $\mathcal{F}$  to  $\{1, 2, \dots, 2n\}$  is  $\mathcal{F}_n$ , while  $\mathcal{C}_n$  arises from  $\mathcal{F}$  by identifying elements which are congruent modulo  $2n$ .

Say that order  $F$  on  $\mathbb{Z}$  is a **generalized infinite fence** if

- the diagram of  $F$  is connected
- at most finitely many elements are comparable to any element of  $F$
- for some natural number  $s$ ,  $F$  has the shift map  $\sigma_s : \mathbb{Z} \rightarrow \mathbb{Z}$  given by  $\sigma_s(i) = i + s$  as an automorphism.

Let  $F$  on  $\mathbb{Z}$  be a **generalized infinite fence** with  $s_0$  a natural number such that  $\sigma_{s_0}$  is an automorphism of  $F$ . Let  $k$  be a natural number such that no element of  $F$  greater than  $ks_0$  is comparable to an element of  $\{1, 2, \dots, s_0\}$ . We call the order  $P$  induced on  $\{1, 2, \dots, ks_0\}$  by  $F$  the **base unit** of  $F$ . Every comparability of  $F$  is recoverable from its base unit. Suppose that  $i, j \in \mathbb{Z}$ , with  $i < j$ . Then  $i <_F j$  if and only if  $i' <_P j'$ , where  $1 \leq i' \leq s_0$ ,  $i' \equiv i \pmod{s_0}$  and  $j' - i' = j - i$ . In the case where  $i'$  and  $j'$  are comparable, it will follow that  $j' \leq ks_0$ , so that  $i', j' \in P$ . One similarly determines whether  $i >_F j$ . It therefore makes sense to call  $F$  **the infinite fence with base unit  $P$  and period  $s_0$** . We write  $F = F(P, s_0)$ .

Suppose that  $F = F(P, s_0)$  is given, with  $|P| = ks_0$ . There are infinitely many other ways of choosing  $s_0$  and  $k$  for  $F$ . Suppose  $t$  is an integer. Let  $\hat{s}_0 = s_0 + ts_0$ ,  $\hat{k} = k + t$ . Then  $\sigma_{\hat{s}_0}$  is an automorphism of  $F$  and no element greater than  $\hat{k}s_0$  is comparable to any element of  $\{1, 2, \dots, \hat{k}s_0\}$ . Choosing  $t = k - 2$ , we get  $\hat{k}s_0 = (k + t)s_0 = (2k - 2)s_0 = 2(s_0 + ts_0) = 2\hat{s}_0$ . Choosing  $s_0$  appropriately, we can thus always pick  $k = 2$ . We shall do this in the remainder of this paper.

Given a generalized infinite fence  $F = F(P, s_0)$  and a natural number  $n$ , the **generalized fence of order  $n$  with base unit  $P$ , period  $s_0$**  is the order  $F_n(P, s_0)$  which is the restriction of  $F$  to  $\{1, 2, \dots, ns_0\}$ . The **generalized crown of order  $n$  with base unit  $P$ , period  $s_0$**  is the order  $C_n(P, s_0)$  on  $\{1, 2, \dots, ns_0\}$  which is obtained from  $F$  by identifying elements which are equivalent modulo  $ns_0$ .

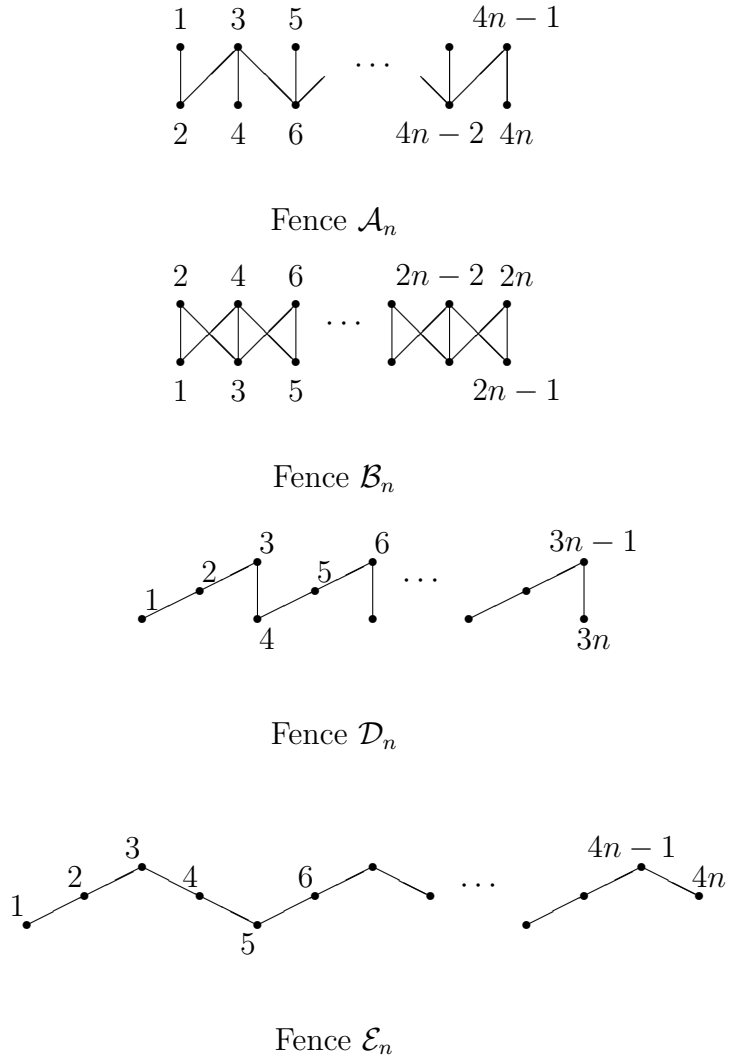


Figure 3: Various generalized fences of order  $n$

**Example 2.1** Consider the ordinary infinite fence  $\mathcal{F}$  depicted in Figure 2. Here  $s_0 = 2$  and  $P$  is the order induced by  $\mathcal{F}$  on  $\{1, 2, 3, 4\}$ . Then  $\mathcal{F} = F(P, 2)$ ,  $\mathcal{F}_n = F_n(P, 2)$ ,  $\mathcal{C}_n = C_n(P, 2)$ . For the orders in Figure 3, we could choose  $s_0 = 4, 2, 3$  and 4 respectively.

### 3 Counting Endomorphisms

Let a generalized infinite fence  $F = F(P, s_0)$ , and a natural number  $n \geq 2$  be given. We will abbreviate  $\sigma_{s_0}$  by  $\sigma$ ,  $C_n(P, s_0)$  by  $C_n$  and  $F_n(P, s_0)$  by  $F_n$ .

**Remark 3.1** The crown  $C_n$  is obtained by identifying elements of  $F$  which are equivalent modulo  $ns_0$ . Let  $i, j \in \{1, 2, \dots, ns_0\}$  be given, with  $i < j$ . Suppose that  $i <_{C_n} j$ . Then  $i' <_F j'$  some  $i', j' \equiv i, j$  modulo  $ns_0$ . Since the base unit of  $F$  is  $P$ , of size  $2s_0$ , we can specify that the difference between  $i'$  and  $j'$  is less than  $2s_0$ . Choosing the lesser of  $i', j'$  to lie in  $\{1, 2, \dots, ns_0\}$ , we can pick  $i', j'$  to lie in  $\{1, 2, \dots, ns_0 + ks_0 = (n+2)s_0\}$ . A similar result holds if  $i >_{C_n} j$ . Thus every comparison in  $C_n$  will be realized if we identify elements of  $F_{n+2}$  which are congruent modulo  $ns_0$ , and  $C_n$  is obtained from identifying the first  $2s_0$  and the last  $2s_0$  elements of  $F_{n+2}$ .

The following are immediate:

**Lemma 3.2** *Endomorphisms of  $C_n$  are in 1-1 correspondence with those homomorphisms  $g$  of  $F_{n+2}$  into  $C_n$  for which*

$$g(i) = g(i + ns_0), \quad i = 1, 2, \dots, 2s_0.$$

**Lemma 3.3** *Homomorphisms  $g$  of  $F_{n+2}$  into  $C_n$  for which  $g(i) = g(i + ns_0)$ ,  $i = 1, 2, \dots, 2s_0$  are in 1-1 correspondence with those homomorphisms  $f$  of  $F_{n+2}$  into  $F$  for which*

$$\begin{aligned} f(1) &\in \{1, 2, \dots, ns_0\} \text{ and} \\ f(i) &\equiv f(i + ns_0) \pmod{ns_0}, \quad i = 1, 2, \dots, 2s_0. \end{aligned}$$

Note that the homomorphisms of  $F_{n+2}$  into  $F$  for which  $f(1) \in \{1, 2, \dots, ns_0\}$  are in  $n$  to 1 correspondence with homomorphisms  $f$  of  $F_{n+2}$  into  $F$  for which  $f(1) \in \{1, 2, \dots, s_0\}$ . This gives the following:

**Lemma 3.4** *Endomorphisms of  $C_n$  are in  $n$  to 1 correspondence with those homomorphisms of  $F_{n+2}$  into  $F$  for which*

$$f(1) \in \{1, 2, \dots, s_0\} \text{ and} \quad (1)$$

$$f(i) \equiv f(i + ns_0) \pmod{ns_0}, \quad i = 1, 2, \dots, 2s_0. \quad (2)$$

Let us consider homomorphisms from  $F_{n+2}$  into  $F$ . Note that  $F_{n+2} = \cup_{i=0}^n \sigma^i(P)$ . Let a homomorphism  $f$  from  $F_{n+2}$  to  $F$  be given. Homomorphism  $f$  corresponds to a sequence of homomorphisms  $\prod_{i=0}^n \{f_i : P \rightarrow F\}$  where

$$f_i(p) = f \circ \sigma^i(p), p \in P. \quad (3)$$

In fact,  $f = \cup_{i=0}^n f_i \circ \sigma^{-i}$ .

**Definition 3.5** Define operators **head** and **tail** from  $F^P$  to  $F^{F_1}$  by

$$\begin{aligned} \mathbf{head}(g) &= g|_{F_1} \\ \mathbf{tail}(g) &= g|_{\sigma(F_1)} \circ \sigma \end{aligned}$$

If  $f_i$  are defined as in (3), then we must have

$$\mathbf{tail}(f_i) = \mathbf{head}(f_{i+1}) \text{ for } i = 0, 1, \dots, n-1. \quad (4)$$

Conversely, if a sequence of homomorphisms  $\prod_{i=0}^n \{f_i : P \rightarrow F\}$  is given satisfying (4), then  $f = \cup_{i=0}^n f_i \circ \sigma^{-i}$  is a well-defined homomorphism from  $F_{n+2}$  to  $F$ . It follows that homomorphisms from  $F_{n+2}$  to  $F$  are in 1-1 correspondence with sequences of homomorphisms  $\prod_{i=0}^n \{f_i : P \rightarrow F\}$  satisfying (4).

If  $f : F_{n+2} \rightarrow F$  also satisfies (1),(2), then  $f$  corresponds to a sequence of homomorphisms  $\prod_{i=0}^n \{f_i : P \rightarrow F\}$  where  $f_0 \equiv f_n$  modulo  $ns_0$ , and thus to a sequence  $\prod_{i=0}^{n-1} \{f_i : P \rightarrow F\}$  where

$$\begin{aligned} \mathbf{tail}(f_i) &= \mathbf{head}(f_{i+1}) \text{ for } i = 0, 1, \dots, n-2, \\ \mathbf{tail}(f_{n-1}) &\equiv \mathbf{head}(f_0) \text{ modulo } ns_0. \end{aligned}$$

**Definition 3.6** Let  $\simeq$  be the equivalence relation on homomorphisms from  $F_1$  to  $F$  given by

$$h \simeq g \text{ if and only if } h = \sigma^k \circ g, \text{ some } k \in \mathbb{Z}.$$

Given any homomorphism  $g$  from  $F_1$  to  $F$ , let  $\bar{g}$  be the unique homomorphism such that  $g \simeq \bar{g}$  and  $\bar{g}(1) \in \{1, 2, \dots, s_0\}$ .

**Remark 3.7** Suppose that  $h \simeq g$ . The  $k$  for which  $h = \sigma^k \circ g$  is unique. Thus if  $h(1) \equiv g(1)$  modulo  $ns_0$  then  $h \equiv g$  modulo  $ns_0$ .

**Definition 3.8** The **type** of a homomorphism  $f : P \rightarrow F$  is the ordered pair

$$\mathbf{type} f = \langle \overline{\mathbf{head}(f)}, \overline{\mathbf{tail}(f)} \rangle.$$

If  $f : F_{n+2} \rightarrow F$  satisfies (1),(2), then  $f$  corresponds to a sequence of homomorphisms  $\prod_{i=0}^{n-1} \{f_i : P \rightarrow F\}$  where

$$\pi_2(\mathbf{type} f_i) = \pi_1(\mathbf{type} f_j), \text{ for } j \equiv i + 1 \text{ modulo } n. \quad (5)$$

Here  $\pi_1, \pi_2$  are the projections.

This has a partial converse; suppose we are given a sequence of homomorphisms  $\prod_{i=0}^{n-1} \{f_i : P \rightarrow F\}$  satisfying (5). Define  $f_n = f_0$ . Let  $\hat{f}_0 = f_0$ . Suppose that homomorphisms  $\hat{f}_i$  have been given for  $i < j \leq n$  so that

$$\begin{aligned} \mathbf{type} \hat{f}_i &= \mathbf{type} f_i, \\ \mathbf{tail} \hat{f}_{i-1} &= \mathbf{head} \hat{f}_i \text{ for } i > 1. \end{aligned}$$

Since  $\mathbf{type} \hat{f}_{j-1} = \mathbf{type} f_{j-1}$  and  $\pi_2(\mathbf{type} f_{j-1}) = \pi_1(\mathbf{type} f_j)$ , we have  $\mathbf{tail} \hat{f}_{j-1} \simeq \mathbf{tail} f_{j-1} \simeq \mathbf{head} f_j$ . Choose  $k$  so that  $\mathbf{tail} \hat{f}_{j-1} = \sigma^k \circ \mathbf{head} f_j$ . Define  $\hat{f}_j = \sigma^k \circ f_j$ . Then  $\mathbf{type} \hat{f}_j = \mathbf{type} f_j$ , while  $\mathbf{tail} \hat{f}_{j-1} = \sigma^k \circ \mathbf{head} f_j = \mathbf{head} \sigma^k \circ f_j = \mathbf{head} \hat{f}_j$ .

This is the unique way to construct homomorphisms  $\hat{f}_i$  such that

$$\begin{aligned} \hat{f}_0(1) &\in \{1, 2, \dots, s_0\} \text{ and} \\ \mathbf{type} \hat{f}_i &= \mathbf{type} f_i, \\ \mathbf{tail} \hat{f}_{i-1} &= \mathbf{head} \hat{f}_i \text{ for } 1 \leq i \leq n. \end{aligned}$$

There is no flexibility in this construction. Thus, while  $\mathbf{type} \hat{f}_n = \mathbf{type} \hat{f}_0$ , so that  $\mathbf{tail} \hat{f}_{n-1} = \mathbf{head} \hat{f}_n \simeq \mathbf{head} \hat{f}_0$ , we cannot guarantee that  $\hat{f}_n \equiv \hat{f}_0$  modulo  $ns_0$ , which would imply that  $\mathbf{tail} \hat{f}_{n-1} \equiv \mathbf{head} \hat{f}_0$  modulo  $ns_0$ .

The homomorphism  $f = \cup_{i=0}^n \hat{f}_i$  is a well-defined homomorphism from  $F_{n+2}$  to  $F$ , but may not satisfy (1),(2). If, however,  $\hat{f}_n(1) \equiv \hat{f}_0(1)$  modulo  $ns_0$ , then we have not just  $\mathbf{head} \hat{f}_n \simeq \mathbf{head} \hat{f}_0$ , but  $\mathbf{head} \hat{f}_n \equiv \mathbf{head} \hat{f}_0$  modulo  $ns_0$ , so that (1),(2) are satisfied.

Suppose that  $r \in \mathbb{N}$  and  $g : F_r \rightarrow F$  is a homomorphism. Assign a weight to  $g$  by

$$w(g) = g(s_0(r-1) + 1) - g(1)$$



If we have homomorphisms  $f_i : P = F_2 \rightarrow F$ ,  $f : F_{n+2} \rightarrow F$  satisfying  $\hat{f}_i(p) = f \circ \sigma^i(p)$ ,  $p \in P$ , or equivalently,  $f = \cup_{i=0}^n \hat{f}_i \circ \sigma^{-i}$ , then

$$\begin{aligned} \sum_{i=0}^n w(\hat{f}_i) &= \sum_{i=0}^n \hat{f}_i(s_0 + 1) - \hat{f}_i(1) \\ &= \sum_{i=0}^n f((i+1)s_0 + 1) - f(is_0 + 1) \\ &= f((n+1)s_0 + 1) - f(is_0 + 1) \\ &= w(f). \end{aligned}$$

In this language, our previous discussion noted that  $f : F_{n+2} \rightarrow F$  satisfies (1),(2) if and only if

$$\begin{aligned} \sum_{i=0}^{n-1} w(\hat{f}_i) &= \hat{f}_{n-1}(s_0 + 1) - \hat{f}_0(1) \\ &= \hat{f}_n(1) - \hat{f}_0(1) \\ &\equiv 0 \text{ modulo } ns_0 \end{aligned}$$

**Definition 3.9** There are finitely many homomorphisms  $h$  of  $P$  into  $F$  for which  $h(1) \in \{1, 2, \dots, s_0\}$ . This is because  $h(i)$  and  $h(i+1)$  can differ by at most  $m-1$ , so that the range of  $h$  is restricted to  $[1 - (m-1)^2, s_0 + (m-1)^2]$ . Say that there are  $r$  possible equivalence classes for **head** ( $h$ ), **tail** ( $h$ ), labelled  $1, 2, \dots, r$ . This gives us a natural labelling of homomorphism types:

**type**  $h \in (i, j)$  if and only if **head**( $h$ )  $\in i$ , **tail**( $h$ )  $\in j$ .

For  $i, j \in \{1, 2, \dots, r\}$ , let  $G_{ij}$  be the set containing those homomorphisms  $h : P \rightarrow F$  for which  $h(1) \in \{1, 2, \dots, s_0\}$  and **type**  $h = (i, j)$ .

Let  $\mathcal{G}_n$  be the set of homomorphisms  $f$  from  $F_{n+2}$  to  $F$  for which (1),(2) hold. Let  $G_n = \cup \prod_{k=0}^{n-1} G_{i_k j_k}$  where the union is taken over sequences  $\prod_{k=0}^{n-1} (i_k, j_k)$  with  $i_k = j_r$  when  $r \equiv k+1$  modulo  $n$ . Assign a weight to elements of  $G_n$  by  $w(\prod_{k=0}^{n-1} f_i) = \sum_{i=0}^{n-1} w(f_i)$ . Let  $\mathcal{S}_n = \{x \in G_n : w(x) \equiv 0 \text{ modulo } ns_0\}$ .

With this notation, our previous discussion is summarized by a theorem:

**Theorem 3.10** *The map*

$$f \rightarrow \prod_{i=0}^{n-1} \{f_i\} \text{ where the } f_i \text{ are given by (3)}$$

*is a bijection between  $\mathcal{G}_n$  and  $\mathcal{S}_n$ .*

**Definition 3.11** If  $S$  is a set of homomorphisms, denote by  $\Phi_S$  the generating homomorphism of  $S$  with respect to  $w$ :

$$\Phi_S(x) = \sum_{h \in S} x^{w(h)}$$

If  $B = [b_{ij}]_{r \times r}$ , then the  $ij^{th}$  entry of  $B^n$  is  $\sum \prod_{k=0}^{n-1} b_{i_k j_k}$  where the sum is taken over sequences  $\prod_{k=0}^{n-1} (i_k, j_k)$  with

$$i_0 = i, j_{n-1} = j \text{ and } j_k = i_{k+1} \text{ for } 0 \leq k \leq r-2.$$

If  $i = j$ , then we also have  $j_{n-1} = i_0$ .

These are just the restrictions on the indices of  $G_n$ . Let  $A = [\Phi_{\mathcal{G}_{ij}}]_{r \times r}$ . The notation  $[y^n]h(y)$  refers to the coefficient of  $y^n$  in a series expansion of  $h$ .

Then

$$\begin{aligned} \phi_{G_n} &= \phi_{\cup \prod_{k=0}^{n-1} G_{i_k j_k}} \\ &= \sum_{k=0}^{n-1} \prod \phi_{G_{i_k j_k}} \\ &= \text{trace}(A^n) \\ &= \text{trace}([y^n] \sum_{k \geq 0} (yA)^k) \\ &= [y^n] \text{trace}((I - yA)^{-1}) \end{aligned}$$

Thus

$$|\mathcal{S}_n| = \sum_{t \equiv 0 \text{ modulo } ns_0} [x^t y^n] \text{trace}((I - yA)^{-1}).$$

**Theorem 3.12** *The number of endomorphisms of  $C_n$  is*

$$n \sum_{t \equiv 0 \text{ modulo } ns_0} [x^t y^n] \text{trace}((I - yA)^{-1}).$$

## 4 Examples

### 4.1 Ordinary Crowns

Consider ordinary crowns  $\mathcal{C}_n$ . Order  $\mathcal{F}_1$  is the 2-element chain with  $1 <_{\mathcal{F}_1} 2$ . If  $f : \mathcal{F}_1 \rightarrow \mathcal{F}$  is a homomorphism with  $f(1) \in \{1, 2\}$ , we have 4 possibilities:

- (1)  $f(1) = 1, f(2) = 2;$
- (2)  $f(1) = 1, f(2) = 0;$
- (3)  $f(1) = 1, f(2) = 1;$
- (4)  $f(1) = 2, f(2) = 2.$

If  $h$  is a homomorphism from  $\mathcal{F}_m$  to  $\mathcal{F}$ , there are thus 4 possible equivalence classes for **head** ( $h$ ), **tail** ( $h$ ). To find the matrix  $A$  of Theorem 3.12 we need to find the  $G_{ij}$ . As an example, consider the set  $G_{12}$ . This set consists of two homomorphisms from  $\mathcal{F}_2$  to  $\mathcal{F}$ , namely  $g_1$  and  $g_2$  where

$$\begin{aligned} g_1(1) &= 1, & g_1(2) &= 2, & g_1(3) &= 1, & g_1(4) &= 0; \\ g_2(1) &= 1, & g_2(2) &= 2, & g_2(3) &= 3, & g_2(4) &= 2. \end{aligned}$$

We see that

$$\begin{aligned} w(g_1) &= g_1(3) - g_1(1) = 1 - 1 = 0 \\ w(g_2) &= g_2(3) - g_2(1) = 3 - 1 = 2. \end{aligned}$$

Thus  $\phi_{G_{12}} = 1 + x^2$ . In this way we determine

$$A = \begin{bmatrix} 1+x^2 & 1+x^2 & 1+x^2 & x \\ x^{-2}+1 & x^{-2}+1 & x^{-2}+1 & x^{-1} \\ 1 & 1 & 1 & 0 \\ x^{-1}+x & x^{-1}+x & x^{-1}+x & 1 \end{bmatrix}.$$

Suppose that  $h$  is a homomorphism from  $\mathcal{F}_{n+2}$  to  $\mathcal{F}$ . It must be the case that  $|h(i) - h(i+1)| \leq 1$  for each  $i$ . Thus  $w(h) \equiv 0$  modulo  $2n$  if and only if  $w(h) = 0$  or  $w(h) = \pm 2n$ . The latter two weights occur exactly in the case when  $h$  corresponds to one of the  $2n$  automorphisms of  $C_n$ .

For ease of computation, we replace  $A$  with  $M = x^2 A$ :

$$M = \begin{bmatrix} x^2+x^4 & x^2+x^4 & x^2+x^4 & x^3 \\ x^2+1 & x^2+1 & x^2+1 & x \\ x^2 & x^2 & x^2 & 0 \\ x^3+x & x^3+x & x^3+x & x^2 \end{bmatrix}.$$

Let  $c_n$  be the number of endomorphisms of a crown on  $2n$  elements. Then

$$c_n = 2n + n[x^{2n}y^n]\text{trace}(I - yM)^{-1}$$

and

$$\text{trace}(I - yM)^{-1} = \frac{4 - 12yx^2 + 2y^2x^4 - 3y - 3x^4y}{1 - 4yx^2 + y^2x^4 - y - x^4y}.$$

This expression is even in  $x$ , so we have

$$\begin{aligned} c_n &= 2n + n[x^n y^n] \frac{4 - 12xy + 2x^2y^2 - 3y - 3x^2y}{1 - 4xy + x^2y^2 - y - x^2y} \\ &= 2n + n[x^n y^n] \frac{1 - x^2y^2}{1 - 4xy + x^2y^2 - y - x^2y} \end{aligned}$$

since the constant term in  $y$  is irrelevant. Letting  $z = xy$ , we now have

$$\begin{aligned} c_n &= 2n + n[z^n] \frac{1 - z^2}{(1 - z)^2 - y(1 + x)^2} \\ &= 2n + [z^n] z \frac{\partial}{\partial z} \frac{1 - z^2}{(1 - z)^2} \left( 1 - y \frac{(1 + x)^2}{(1 - z)^2} \right)^{-1} \\ &= 2n + [z^n] z \frac{\partial}{\partial z} \frac{1 + z}{1 - z} \sum_{i \geq 0} \frac{y^i (1 + x)^{2i}}{(1 - z)^{2i}} \\ &= 2n + [z^n] z \frac{\partial}{\partial z} \frac{1 + z}{1 - z} \sum_{i \geq 0} \binom{2i}{i} \left( \frac{z}{(1 - z)^2} \right)^i \end{aligned}$$

upon extracting  $[x^i](1 + x)^{2i}$ , and using the fact that  $\sum_{i \geq 0} \binom{2i}{i} t^i = (1 - 4t)^{-1/2}$ , we obtain

$$\begin{aligned} c_n &= 2n + [z^n] z \frac{\partial}{\partial z} \frac{1 + z}{1 - z} \left( 1 - \frac{4z}{(1 - z)^2} \right)^{-1/2} \\ &= 2n + [z^n] z \frac{\partial}{\partial z} \frac{1 + z}{\sqrt{(1 - z)^2 - 4z}} = 2n + [z^n] z \frac{\partial}{\partial z} (1 + z) a^{-1/2} \end{aligned}$$

where  $a = 1 - 6z + z^2$ . Finally, we simplify to obtain

$$\begin{aligned} c_n &= [z^n] \left\{ \frac{2z}{(1 - z)^2} + z(a^{-1/2} - \frac{1}{2}(1 + z)a^{-3/2}(-6 + 2z)) \right\} \\ &= [z^n] \left\{ \frac{2z}{(1 - z)^2} + \frac{4z(1 - z)}{a^{3/2}} \right\} = [z^n] \frac{2z(a^{3/2} + 2(1 - z)^3)}{a^{3/2}(1 - z)^2}, \end{aligned}$$

agreeing with [1, p. 141]

## 4.2 Zippers

Those not familiar with generating functions may prefer enumerations where the result is expressed in terms of sums of products of choice functions. The results of [6, 7, 3] are given in this form. Of course, such an expression is easily obtained from a generating function. In the case of ordinary crowns, our previous example, it was extra work to derive a generating function. In the present section we will leave our result in terms of choice functions.

The **zipper**  $Z_n$  on  $2n$  elements is obtained by identifying elements of  $\mathcal{B}_{n+1}$  which are congruent modulo  $2n$  (See Figure 3). Let  $z_n$  be the number of endomorphisms of a zipper on  $2n$  elements. We get

$$A = \begin{bmatrix} x^{-2} + 1 + x^2 & x^{-2} + 1 & 1 + x^2 & 1 & x \\ 1 + x^2 & 1 & 1 + x^2 + x^4 & 1 & x^3 \\ x^{-2} + 1 & x^{-4} + x^{-2} + 1 & 1 & 1 & x^{-1} \\ 1 & 1 & 1 & 1 & 0 \\ x^{-1} & x^{-3} & x & 0 & 1 \end{bmatrix}$$

and  $z_n = n[(x^{-2n} + x^0 + x^{2n})y^n]\text{trace}(I - yA)^{-1}$ . If  $x$  is replaced by  $x^{-1}$  in  $A$ , we obtain the transpose of  $A$ , so  $[x^{-2n}y^n]\text{trace}(I - yA)^{-1} = [x^{2n}y^n]\text{trace}(I - yA)^{-1}$ . Using this observation and multiplying the entries of  $A$  by  $x^2$  (in order to obtain a formal power series), we have

$$z_n = n\{[x^{2n}] + 2[x^0]\}[y^n]\text{trace}(I - yx^2A)^{-1}.$$

We can obtain an expression for  $\text{trace}(I - yx^2A)^{-1}$ . It is an even expression in  $x$  and if we perform a partial fraction expansion and ignore the constant term in  $y$  we obtain

$$z_n = n\{[x^n] + 2[x^0]\}[y^n](A + 2B)$$

where

$$A = \frac{1 + 2x^2y^2}{1 + y - xy + x^2y - 2x^2y^2},$$

$$B = \frac{1 - y - 2xy - x^2y}{1 - 2y - 4xy - 2x^2y + y^2 + 2xy^2 - x^2y^2 + 2x^3y^2 + x^4y^2}.$$

It is straightforward to check that  $[x^0y^n]A = (-1)^n$  and  $[x^0y^n]B = 1$ . Letting  $z = xy$ , we now have

$$[x^n y^n]A = [x^n y^n] \frac{1 + 2z^2}{1 + z - 2z^2 + y(1 - x)^2}$$

$$\begin{aligned}
&= [z^n] \frac{1+2z^2}{1+z-2z^2} \left( 1 + \frac{y(1-x)^2}{1+z-2z^2} \right)^{-1} \\
&= [z^n] (1+2z^2) \sum_{i \geq 0} (-1)^i \frac{y^i (1-x)^{2i}}{(1+z-2z^2)^{i+1}} \\
&= [z^n] (1+2z^2) \sum_{i \geq 0} \binom{2i}{i} z^i (1-z)^{-(i+1)} (1+2z)^{-(i+1)}
\end{aligned}$$

upon extracting  $[x^i](1+x)^{2i}$ . Using similar techniques and letting  $w = y(1+x)^2$ , we can also show that

$$\begin{aligned}
[x^n y^n] B &= [z^n] \frac{1-w}{1-3z^2-2w+w^2-2zw} \\
&= [z^n] \frac{1}{1-w} \left( 1 - \frac{z(2w+3z)}{(1-w)^2} \right)^{-1} \\
&= [z^n] \sum_{i \geq 0} \frac{z^i (2w+3z)^i}{(1-w)^{2i+1}}.
\end{aligned}$$

Extracting the above coefficients and putting the terms together, we conclude that

$$\begin{aligned}
z_n = n \Big\{ & 4 + 2(-1)^n + \sum_{i,j} \binom{2i}{i} \binom{i+j}{j} \left\{ \binom{n-j}{i} + 2 \binom{n-j-2}{i} \right\} (-2)^j \\
& + \sum_{i,j} \binom{i}{j} \binom{n}{2i} \binom{2n-4i+2j}{n-2i+j} 2^{j+1} 3^{i-j} \Big\}.
\end{aligned}$$

## 5 Concluding Remarks and Numerical Data

There is another way to exploit the equation  $\phi_{G_n} = \text{trace}(A^n)$  that deserves mention. It may be that the matrix  $A$  is diagonalizable, so that we can write  $A = SDS^{-1}$ ,  $D$  diagonal. This is the case, for example, with ordinary crowns. In this case, we have

$$\phi_{G_n} = \text{trace}(SD^nS^{-1}).$$

Since  $D$  is diagonal, we can get a closed form for  $D^n$ , and hence  $\phi_{G_n}$  as a function of  $n$ . In any case, the current formulation  $\phi_{G_n} = \text{trace}(A^n)$  allows easy computation of the number of maps for specific  $n$  for the generalized crowns corresponding to any of the examples in Figure 3. For  $n \geq 2$ , let  $D_n$  be the generalized crown

Table 1: Numbers of endomorphisms of  $Z_n$  and  $D_n$ .

$n$	$z_n$	$d_n$
2	275	139
3	951	1,646
4	4,868	22,075
5	31,735	310,442
6	252,054	4,471,966
7	1,980,727	65,398,070
8	15,463,416	966,609,787
9	119,914,191	14,401,689,461
10	924,752,690	215,922,873,094
11	7,097,502,159	3,253,709,282,423
12	54,253,458,780	49,234,244,569,030
13	413,281,739,949	747,605,163,039,752
14	3,138,868,642,826	11,385,905,901,377,440

corresponding to  $\mathcal{D}_n$ . Thus  $D_n$  has the six element base unit  $1 < 2 < 3 > 4 < 5 < 6$  as in Figure 3. Let  $d_n$  be the number of endomorphisms of  $D_n$ . We give values for  $z_n$  and  $d_n$  in Table 1 .

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